If we choose the right potential function $V_N(r)$, then the wave function for the whole nucleus can be written as a product of the single particle wave functions for all *A* nucleons, or at least schematically:

$$\Psi_{Nucleus}(\vec{r}) = \prod_{i=1}^{A} \psi_{nlm}(\vec{r}_i)$$

oversimplification here... actually, it has to be written as an antisymmetrized product wavefunction since the nucleons are identical Fermions – the procedure is well-documented in advanced textbooks in any case!

With total angular momentum given by:

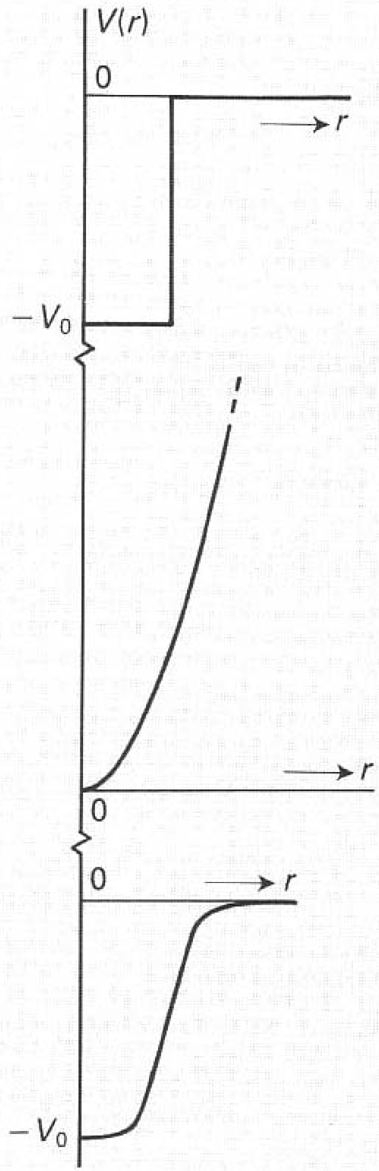
$$\vec{J} = \sum_{i=1}^{A} \vec{j}_i\,, \qquad \vec{j}_i = \vec{\ell}_i + \vec{s}_i\,, \qquad (s = \tfrac{1}{2})$$

And parity:

$$\pi = \prod_{i=1}^{A} (-1)^{\ell_i}$$

Always + for an even number of nucleons...

What to use for $V_N(r)$? – three candidate potential functions:



Square well

$r<R$: $V(r) = -V_0$

$r>R$: $V(r) = 0$

Advantage: easy to write down
Disadvantages:
numerical solutions only
edges unrealistically sharp

Harmonic oscillator

$$V(r) = \tfrac{1}{2}M\omega^2 r^2$$

Advantage: easy to write down and can be solved analytically
Disadvantage: potential should not go to infinity, have to cut off the function at some finite r and adjust parameters to fit data.

Saxon–Woods

$$V(r) = \frac{-V_0}{1+\exp\left(\frac{r-R}{d}\right)}$$

Advantage: same shape as measured charge density distributions of nuclei. smooth edge makes sense

Disadvantage:
numerical solution needed

- since both potentials are spherically symmetric, the only difference is in the radial dependence of the wave functions

- amazingly, when parameters are adjusted to make the average potential the same, as shown in the top panel, there is remarkably little difference in the radial probability densities for these two potential energy functions!

- this being the case, the simplicity of the harmonic oscillator potential means that it is strongly preferred as a model for nuclei

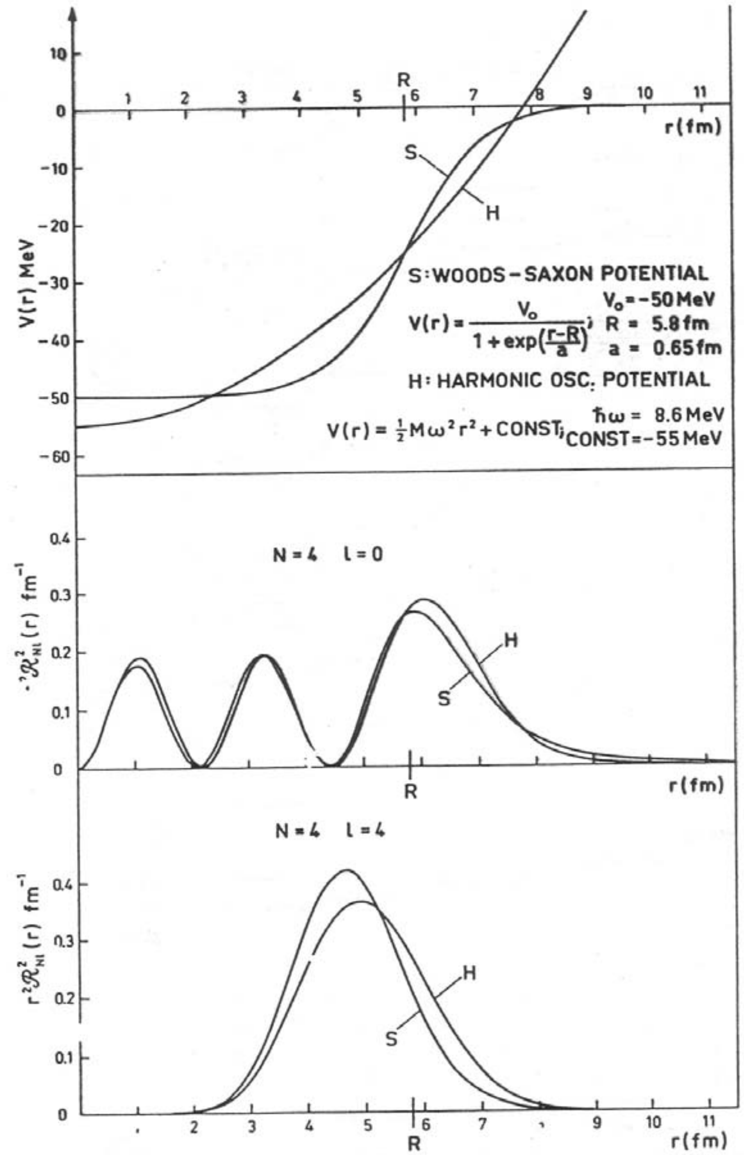


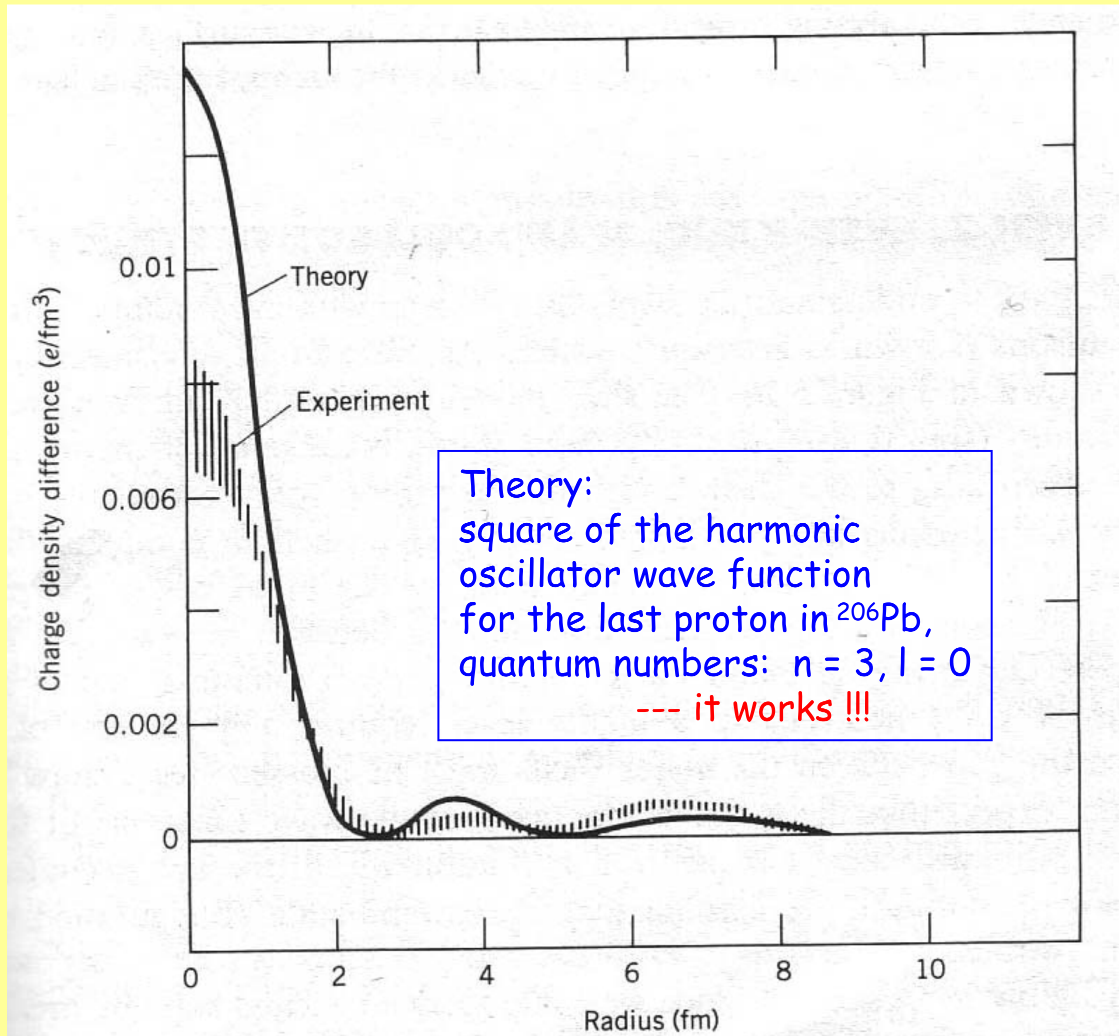
Figure 2-22 The square of the wave function times r^2 for the harmonic oscillator and the Woods-Saxon potential are plotted in units of fm^{-1}.

Evidence that this works:

electric charge density, measured via electron scattering:

$$\rho(\vec{r}) = e \sum_{i=1}^{Z} |\psi_i(\vec{r})|^2$$

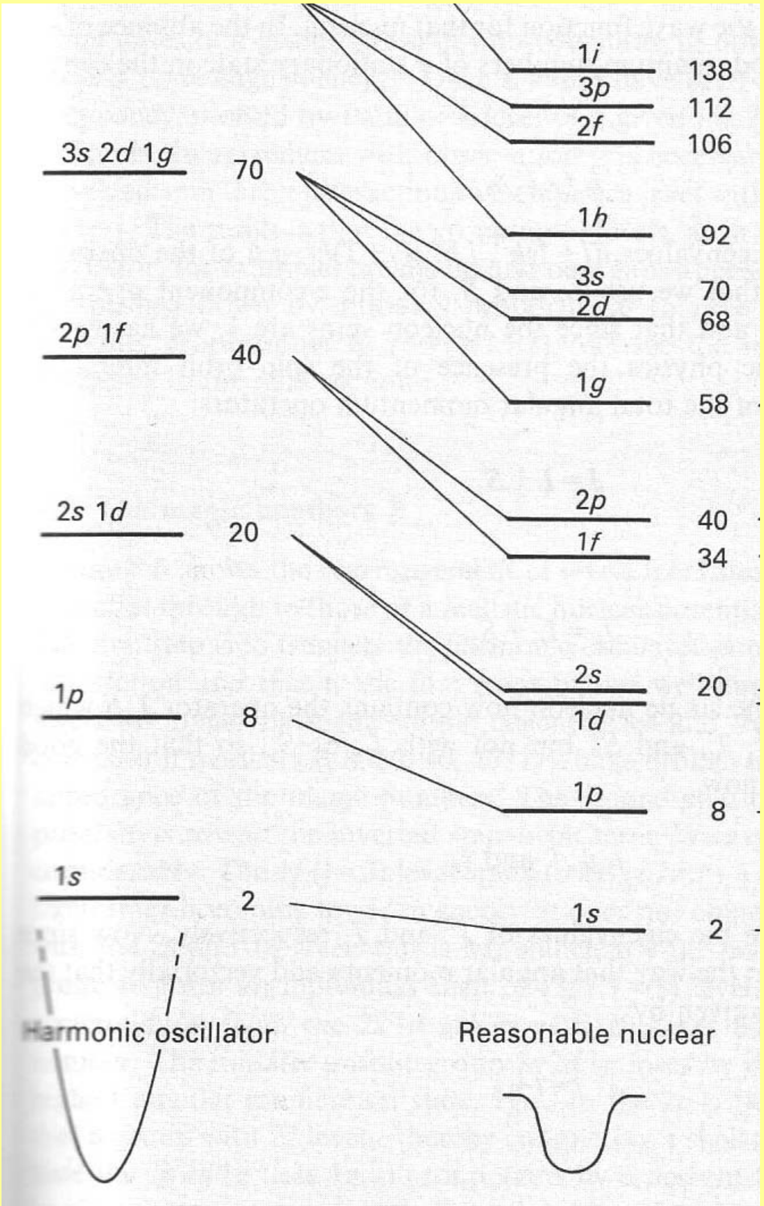
charge density difference between ^{205}Tl and ^{206} Pb is proportional to the square of the wave function for the extra proton in ^{206} Pb, i.e. we can actually measure the square of the wave function for a single proton in a complex nucleus this way!



Theory:
square of the harmonic oscillator wave function for the last proton in ^{206}Pb, quantum numbers: n = 3, l = 0

--- it works !!!

Various potential shapes lead to similar patterns of energy gaps, e.g.:



But the magic numbers are wrong ☹ !

N/Z = 2, 8, 20, 28, 50, 82, 126

Something else is needed to explain the observed behaviour...

- Meyer and Jensen, 1949: enormous breakthrough at the time because it was the only explanation for the observed pattern of "magic numbers" and paved the way for a "periodic table" of nuclei ... and the Nobel prize in physics, 1963!

(http://www.nobel.se/physics/laureates/1963/index.html – see Maria Goeppert-Meyer's Nobel Lecture link on this page)

- simple idea:

$$V_N(r) \Rightarrow V_N(r) + V_{so}(r)\left\langle \vec{\ell} \bullet \vec{s} \right\rangle$$

expectation value

as in the calculation of magnetic moments, we can write:

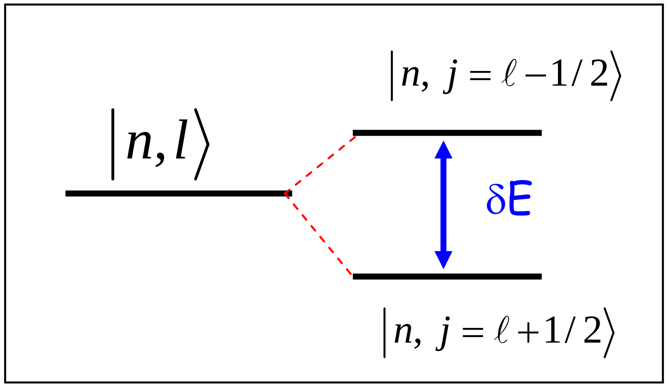
$$\left\langle \vec{\ell} \bullet \vec{s} \right\rangle = \frac{1}{2}\left\langle j^2 - \ell^2 - s^2 \right\rangle = \frac{1}{2}\left(j(j+1) - \ell(\ell+1) - s(s+1)\right)$$

- but there are only two ways the orbital and spin angular momentum can add for a single particle nucleon state:

a) "stretched state" $j = l + \frac{1}{2}$: $\left\langle \vec{\ell} \bullet \vec{s} \right\rangle = \ell s = \frac{\ell}{2}$

b) "jack-knife state" $j = l - \frac{1}{2}$: $\left\langle \vec{\ell} \bullet \vec{s} \right\rangle = -\ s\,(\ell+1) = \frac{-(\ell+1)}{2}$

- The energy shift due to the spin-orbit interaction is between states of the same l but different *j*:

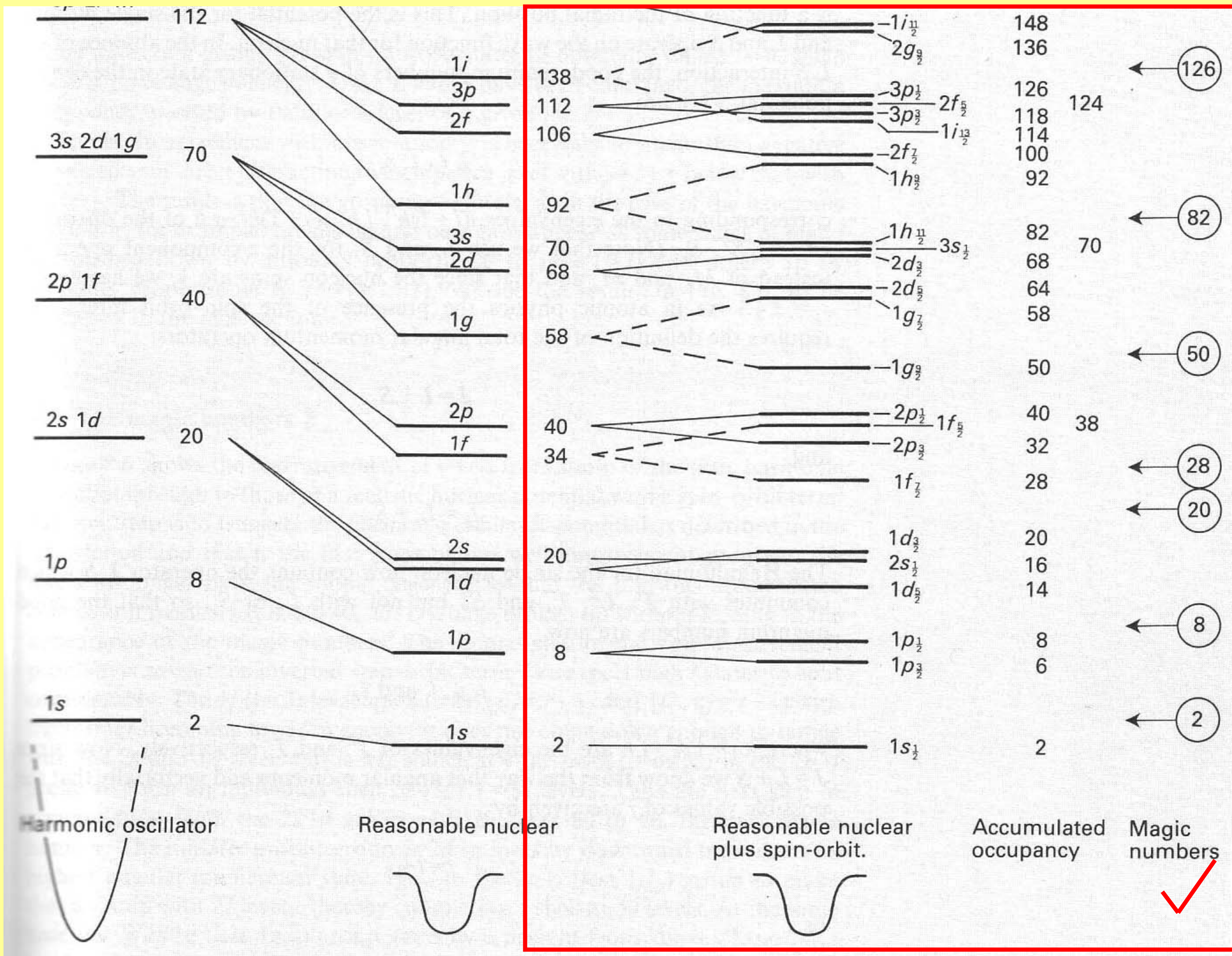


- the splitting is proportional to l and so it increases as the energy increases for the single particle solutions to V(r)

- each state can accommodate (2j+1) neutrons or protons, each with different m_j

- empirically, the sign of the spin-orbit term for nuclei is opposite to that for atoms and the effect is much stronger in nuclei – the phenomenon has nothing to do with magnetism, which is the origin of this effect in atoms, but rather it reflects a basic feature of the strong nuclear force.

- with these features, the spin-orbit potential is the "missing link" required to correctly predict the observed sequence of magic numbers in nuclear physics ☺



Harmonic oscillator
Reasonable nuclear
Reasonable nuclear plus spin-orbit.
Accumulated occupancy
Magic numbers
1s 2
1p 8
2s 1d 20
2p 1f 40
3s 2d 1g 70
112
1s 1p 1d 2s 1f 2p 1g 2d 3s 1h 2f 3p 1i
2 8 20 34 40 58 68 70 92 106 112 138
$1s_{\frac{1}{2}}$ $1p_{\frac{3}{2}}$ $1p_{\frac{1}{2}}$ $1d_{\frac{5}{2}}$ $2s_{\frac{1}{2}}$ $1d_{\frac{3}{2}}$ $1f_{\frac{7}{2}}$ $2p_{\frac{3}{2}}$ $1f_{\frac{5}{2}}$ $2p_{\frac{1}{2}}$ $1g_{\frac{9}{2}}$ $1g_{\frac{7}{2}}$ $2d_{\frac{5}{2}}$ $2d_{\frac{3}{2}}$ $3s_{\frac{1}{2}}$ $1h_{\frac{11}{2}}$ $1h_{\frac{9}{2}}$ $2f_{\frac{7}{2}}$ $1i_{\frac{13}{2}}$ $3p_{\frac{3}{2}}$ $2f_{\frac{5}{2}}$ $3p_{\frac{1}{2}}$ $2g_{\frac{9}{2}}$ $1i_{\frac{11}{2}}$
2 6 8 14 16 20 28 32 38 40 50 58 64 68 70 82 92 100 114 118 124 126 136 148
2 8 20 28 50 82 126

8

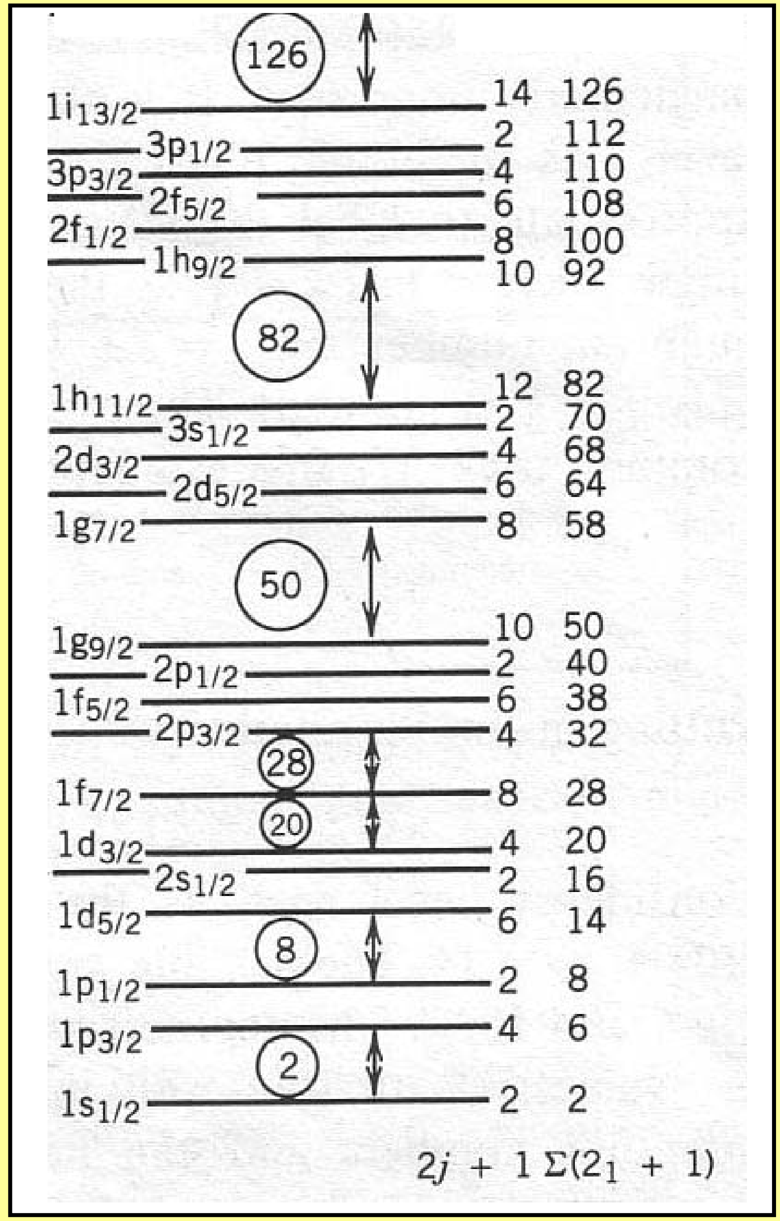
Generic pattern of single particle states solved in a Woods-Saxon (rounded square well) potential model with appropriate spin-orbit interaction to reproduce the observed "magic number" pattern:

State labels: $n\,\ell_{\,j}$

where n labels the order of occurrence of a given l value, and the state labels for orbital angular momentum are:

0	1	2	3	4	5	6
s	p	d	f	g	h	i

Each state can hold (2j +1) neutrons and (2j+1) protons, corresponding to 2(2j+1) distinct configurations of identical nucleons (m_t, m_j) to satisfy the Pauli exclusion principle



First of all, consider a **"closed shell"**, which corresponds to a completely filled single-particle state, e.g. $1s_{1/2}$, $1p_{3/2}$, etc... containing (2j+1) protons or neutrons:

The total angular momentum is:

$$\vec{J} = \sum \vec{j}_i \,, \qquad M = \sum m_i = 0$$

all have the same j in a given shell

each m value is different, ranging from –j to +j

There is effectively only one configuration here, with total z-projection M = 0. Therefore, the total angular momentum of a closed shell must be J = 0!

The total parity is:

$$\pi = \prod (-1)^{\ell} = \left[(-1)^{\ell} \right]^{2j+1} = +$$

Since (2j+1) is always even, the parity of a closed shell is always positive.

Valence Nucleons:

For a **closed shell + n** nucleons, the angular momentum and parity is determined by the **n "valence" nucleons**, since the closed shell contributes $J^\pi = 0^+$:

$$\vec{J} = \sum_{i=1}^{n} \vec{j}_i \ , \qquad \pi = \left[(-1)^{\ell}\right]^{n} = (-1)^{n\ell}$$

The parity is uniquely determined, but there may be several different values of J that are consistent with angular momentum coupling rules. Residual interactions between the valence nucleons in principle determine which of the allowed J has the lowest energy – we can't predict this a priori but can learn from experiment.

"Holes" -- for a state that is almost full, it is simpler to consider angular momentum coupling for the missing nucleons than for the ones that are present:

result for a closed shell

$$\vec{0} = \sum_{i=1}^{n} \vec{j}_i + \sum_{i=(n+1)}^{2j+1} \vec{j}_i$$

magnitudes of the two partial sums have to be the same, and M values opposite.

Total angular momentum for a collection of missing particles, i.e. holes, is the same as for that same collection of particles in a given shell model state.

Magnetic Moments:

As in *lecture 20*, we can write:

$$\mu \equiv g_J \, J \, \mu_N = \left\langle \vec{\mu} \bullet \vec{J} \right\rangle \frac{1}{(J+1)}$$

Predictions are only simple in the limit of one valence nucleon, and then we have:

valence proton:

$$\vec{\mu} = \left(g_{s,p} \, \vec{s} + \vec{\ell} \right) \mu_N$$

valence neutron:

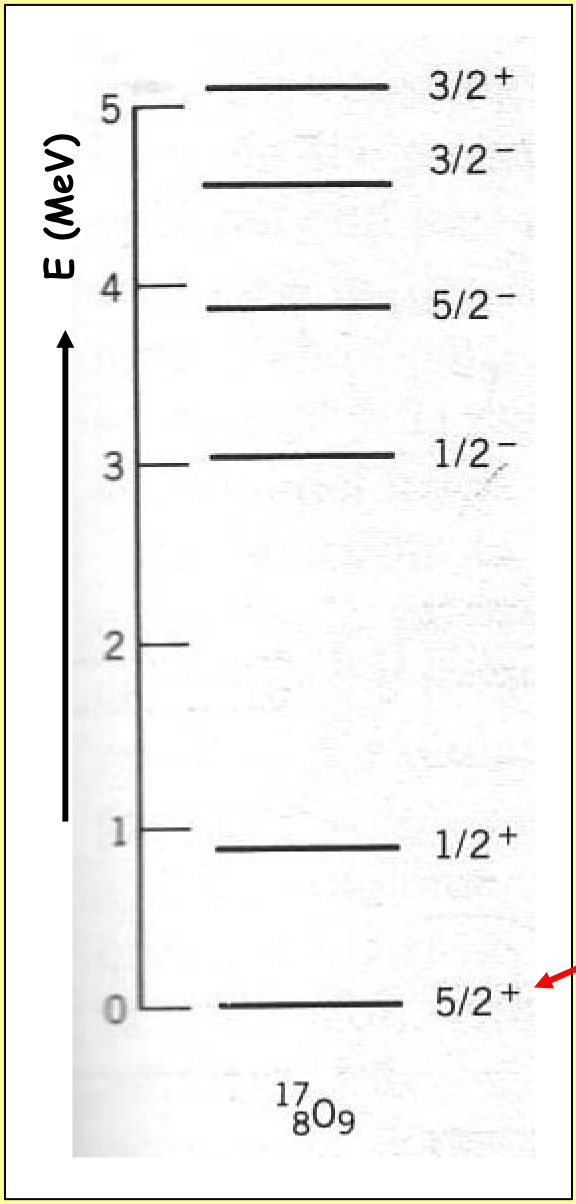
$$\vec{\mu} = g_{s,n} \, \vec{s} \, \mu_N$$

Simplifying the results, we find:

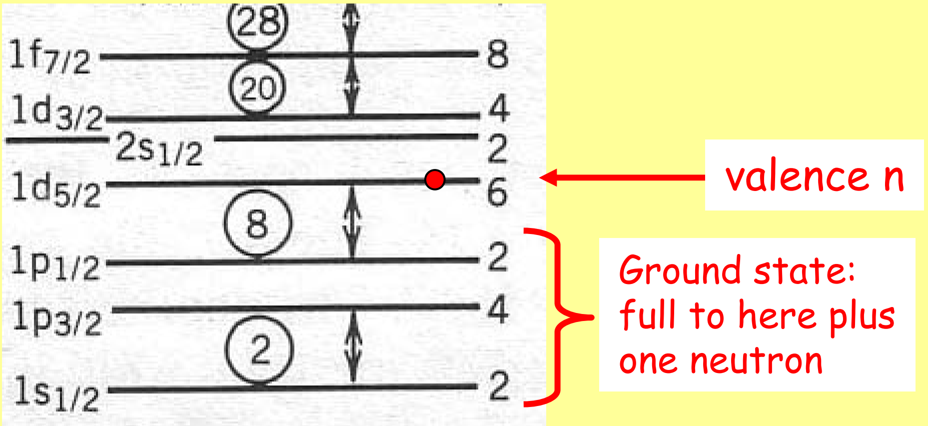
$$j = \ell + 1/2 \qquad \mu = \left[g_\ell \, (j - 1/2) + g_s / 2 \right] \mu_N$$

$$j = \ell - 1/2 \qquad \mu = \left[g_\ell \, j\,(j + 3/2)/(j+1) - g_s \, j/2(j+1) \right] \mu_N$$

with $g_{sp} = +5.58, \quad g_{sn} = -3.83, \quad g_{\ell p} = 1, \quad g_{\ell n} = 0$



There are 8 protons and 9 neutrons, so we only need the low lying states in the shell model spectrum to understand the energy levels:

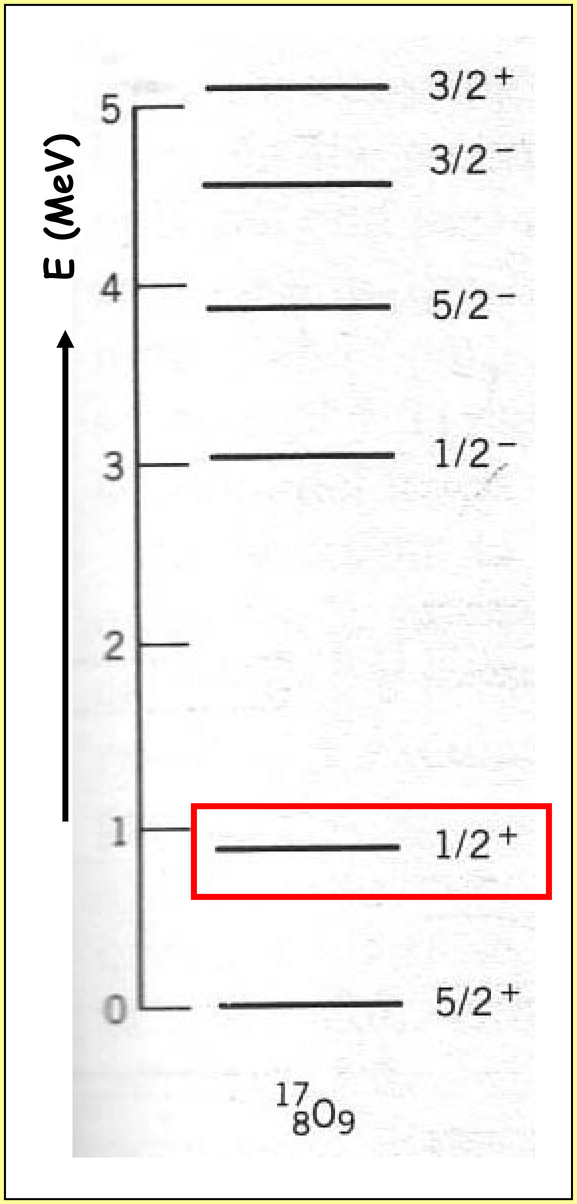


Ground state quantum numbers should be those of the valence neutron in the $1d_{5/2}$ state:
$J^{\pi} = 5/2^{+}$ ☺

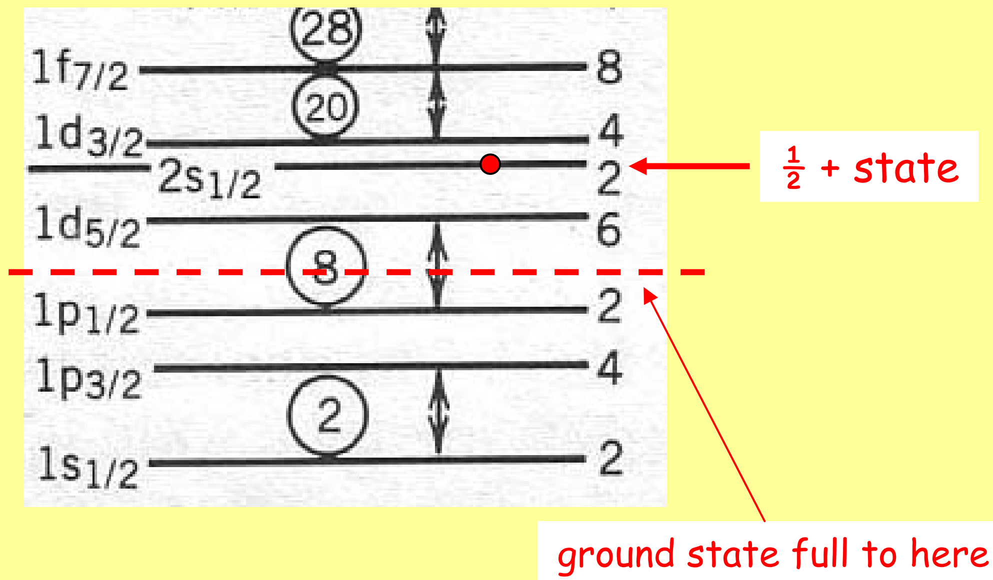
Magnetic moment prediction: $j = l + \frac{1}{2}$, odd neutron
→ $\mu = \mu_{neutron} = -1.91\ \mu_N$

measured value: $-1.89\ \mu_N$ excellent agreement!

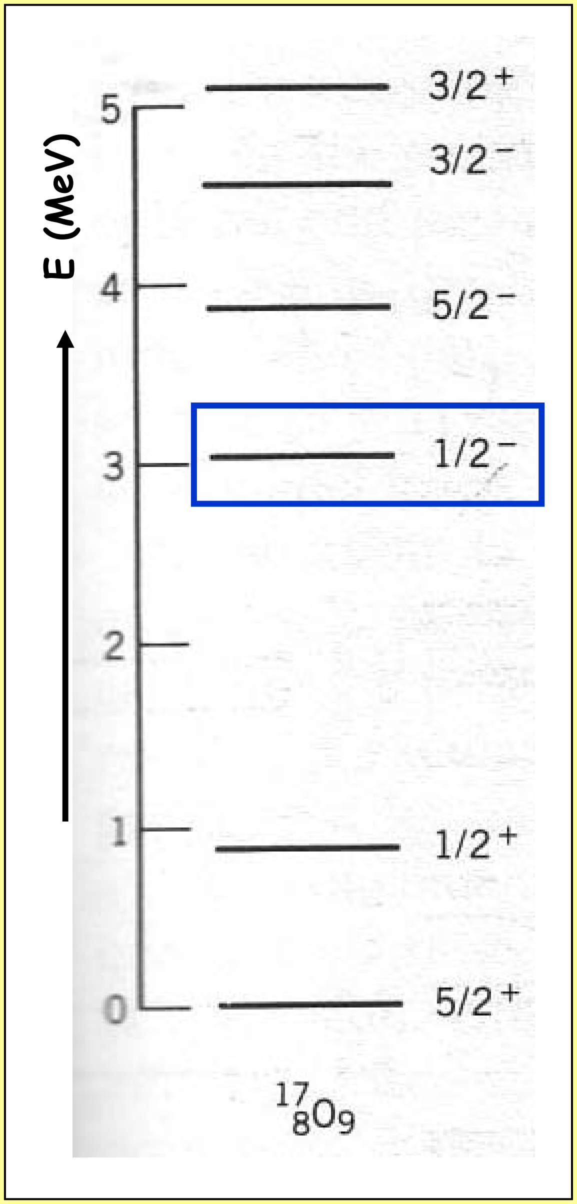
Excited states of ^{17}O can be understood by promoting the valence neutron:



First excited state: $J^{\pi} = \frac{1}{2}^{+}$

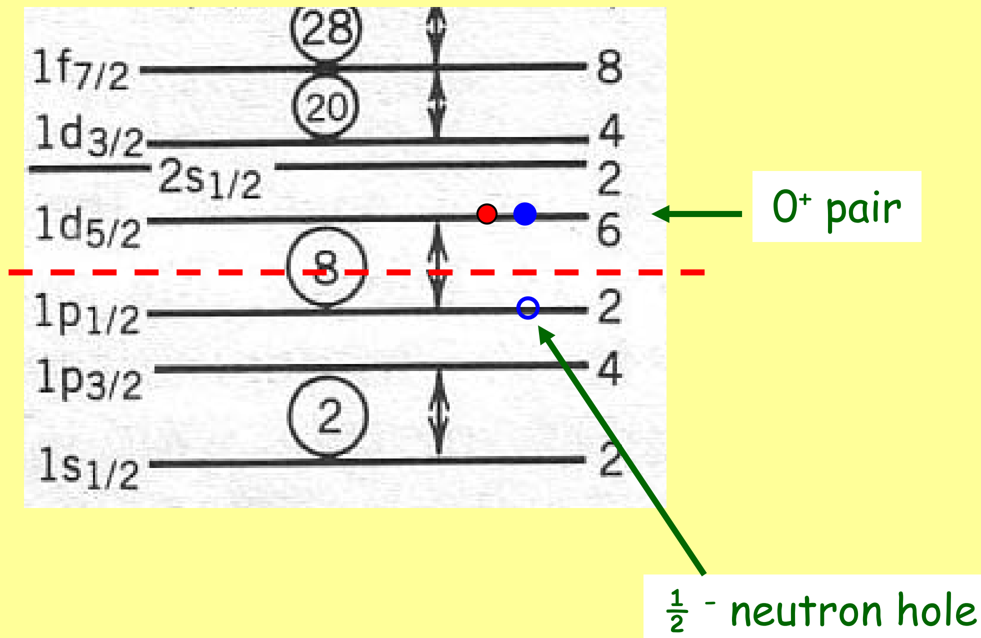


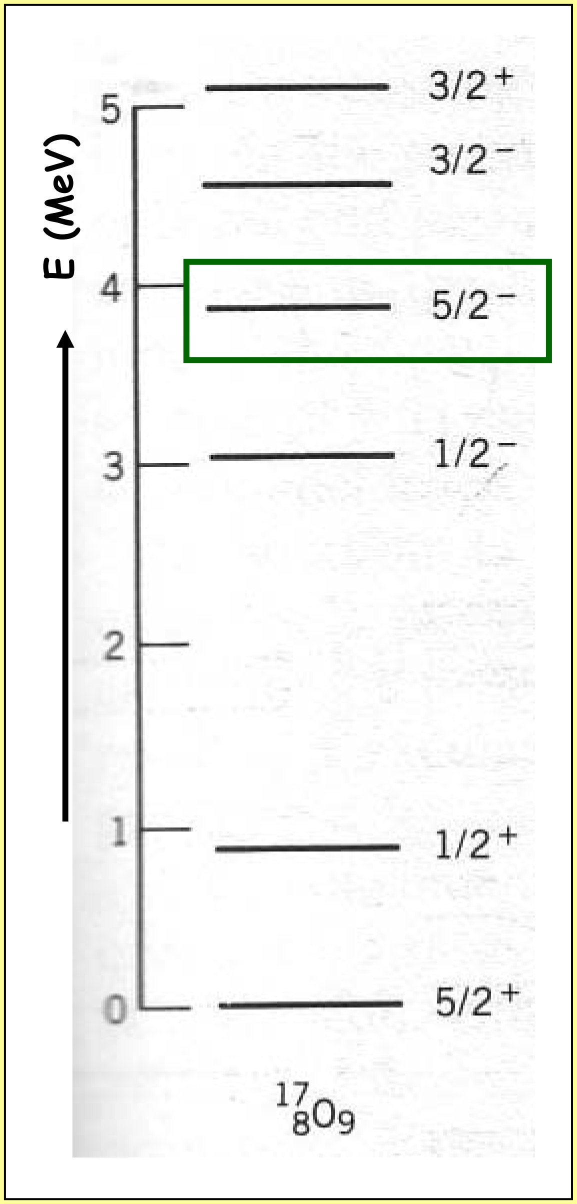
Excited states of ^{17}O:



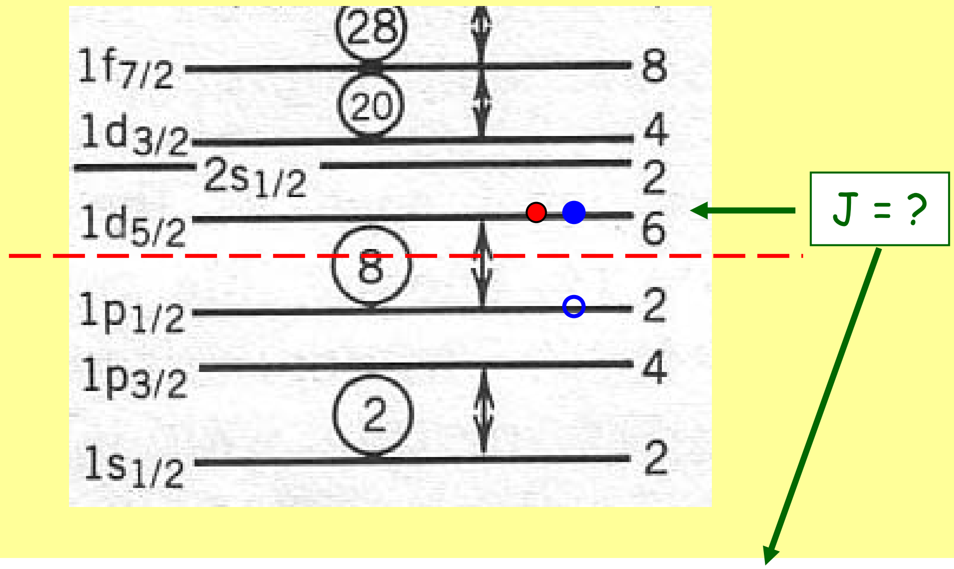
Next excited state: $J^\pi = \frac{1}{2}^-$

→ explained by promoting a neutron from the filled $1\,p_{1/2}$ level to the $1d_{5/2}$ level





The $5/2^-$ state is not so easy: to have negative parity, there must be an odd nucleon in a p state (or f state, but that is higher)



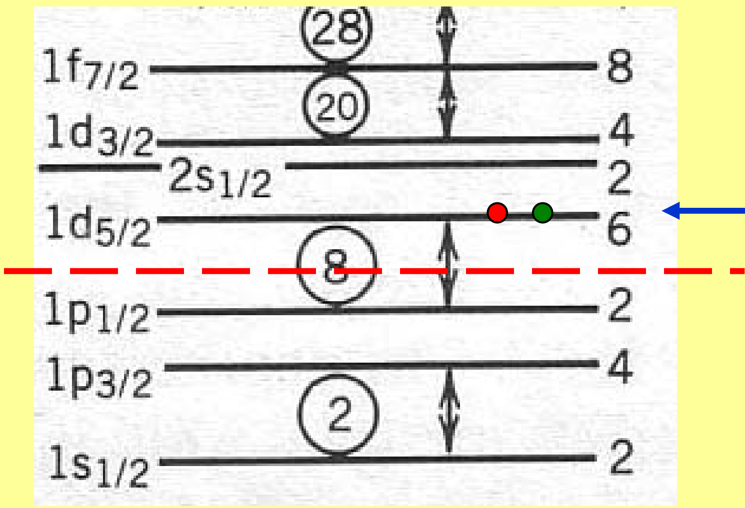
$$\vec{5}/2 + \vec{5}/2 = \vec{J}\,, \quad J = 0,\ 1,\ 2,\ 3,\ 4,\ 5$$

But two neutrons are required to have different values of m_j by the Pauli principle. Writing out the allowed configurations → **only J = 0, 2, 4 are allowed!** (J = 2 will work here)

This problem is much more complicated! The inner "core" nucleons couple to $J^{\pi} = 0^{+}$ but in general there is more than one possibility for the angular momentum coupling of the valence particles.

a) (pp) or (nn) case: Z and N are both even in these cases, so we know that the ground state configuration will be 0^{+} no matter what shell model state they are in. Excited states will have higher angular momentum, with possibilities restricted by the Pauli principle.

b) (np) case: Z and N are both odd in this case. Only 6 examples in the whole nuclear chart!!! In isolation, (np) prefers to form a bound state – the deuteron – with $J^{\pi} = 1^{+}$.



Example: ^{18}F, Z = 9, N = 9

np pair: no restrictions on total J.
possibilities are 0, 1, 2, 3, 4, 5

Ground state of ^{18}F is 1+ i.e. deuteron quantum numbers!

N.B. valence nucleons interact with each other, or all J values would be degenerate!

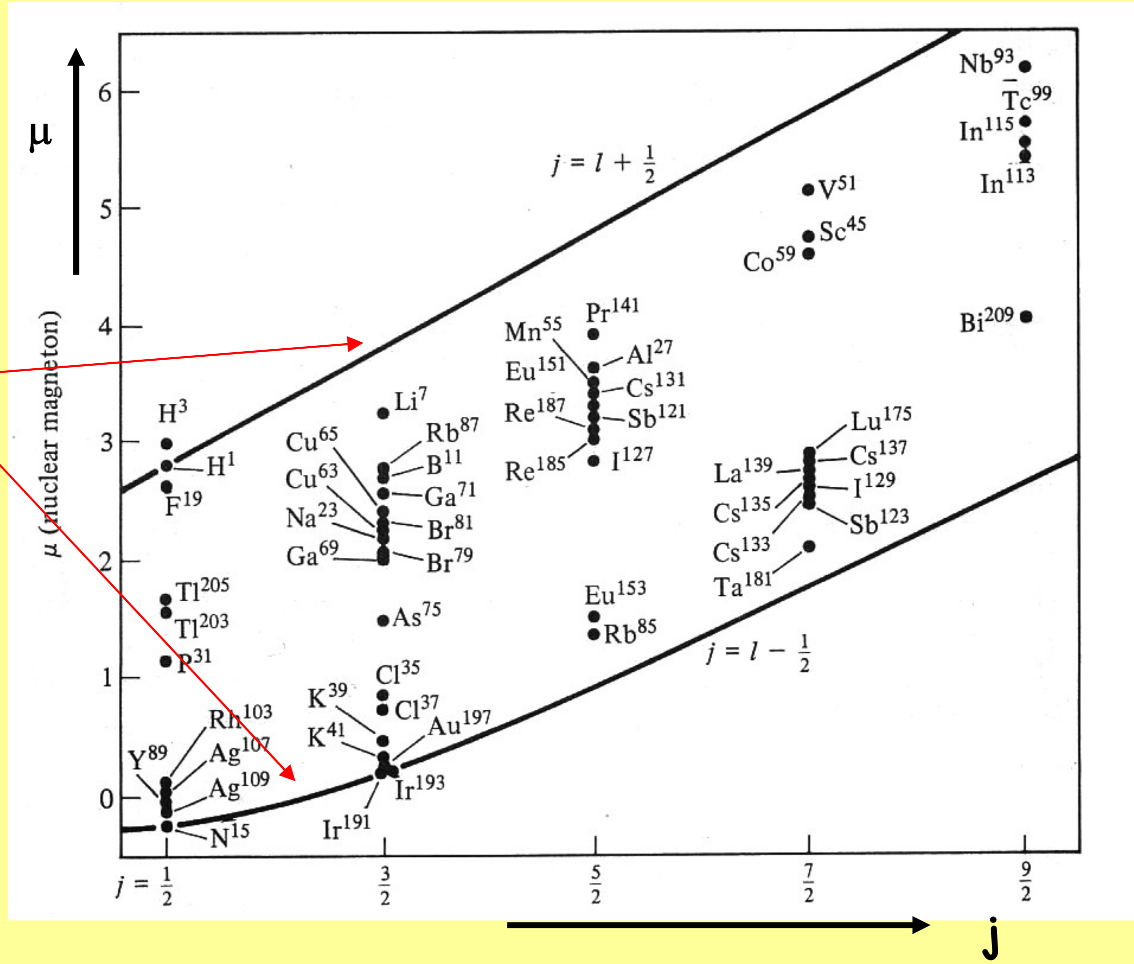
Single particle predictions:

$$j = \ell + 1/2 \qquad \mu = \left[g_\ell \; (j - 1/2) + g_s / 2 \right] \mu_N$$

$$j = \ell - 1/2 \qquad \mu = \left[g_\ell \; j\,(j + 3/2)/(j+1) - g_s \; j/2(j+1) \right] \mu_N$$

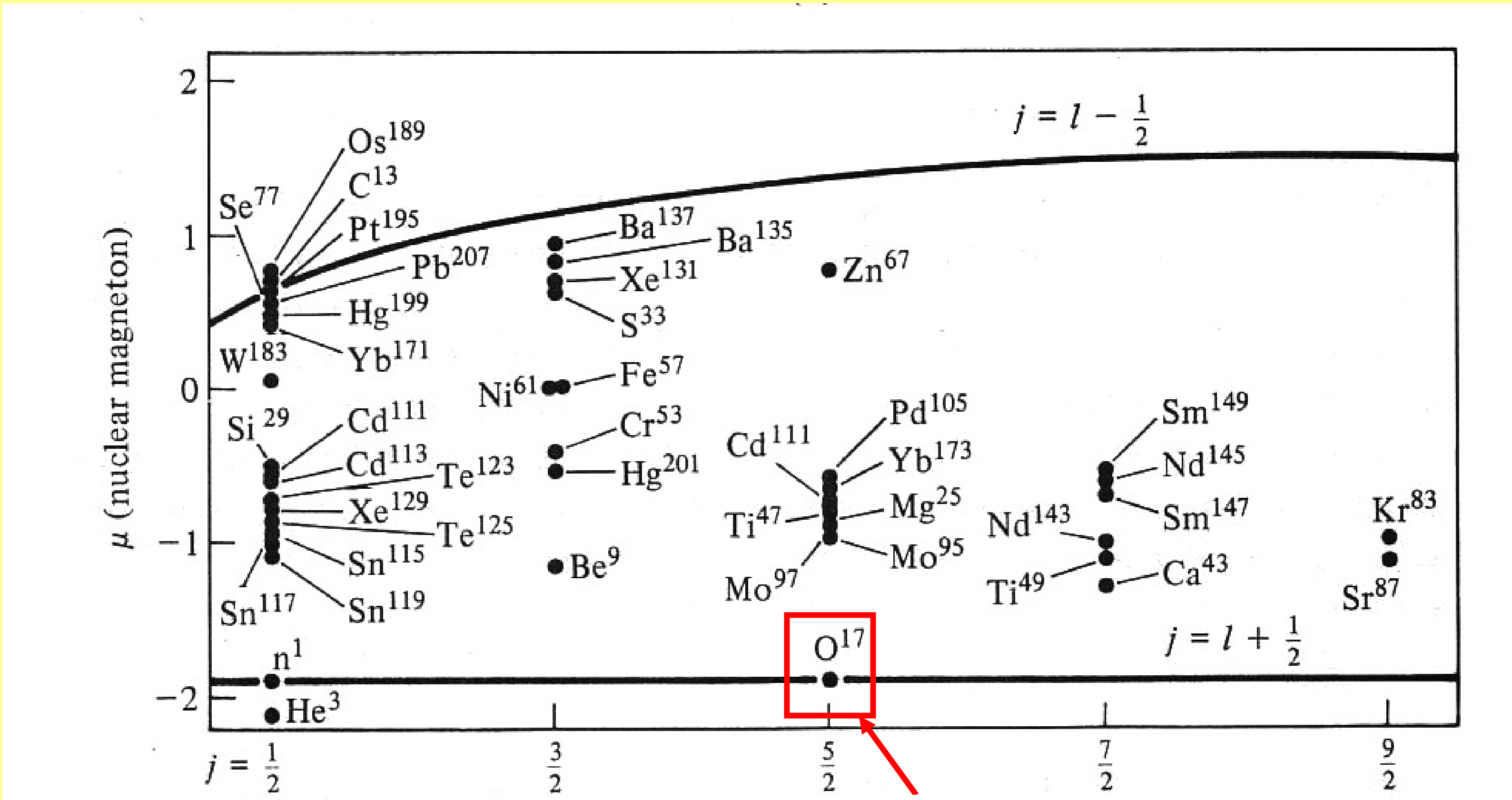
Data for **odd-proton** nuclei:

(lines are calculated from the formulae)



Agreement is not terrific, but the values lie within the two "Schmidt lines"

Odd neutron nuclei:



^{17}O example: "perfect"

What is wrong?

- the single particle model is too simple – nucleons interact with each other
- configurations may be mixed, i.e. linear combinations of different shell model states
- magnetic moments of bound nucleons may not be the same as those of free nucleons…

Title page of a research monograph, Oxford, 1990:

Modern Theories of Nuclear Moments

B. CASTEL
Queen's University, Kingston, Canada

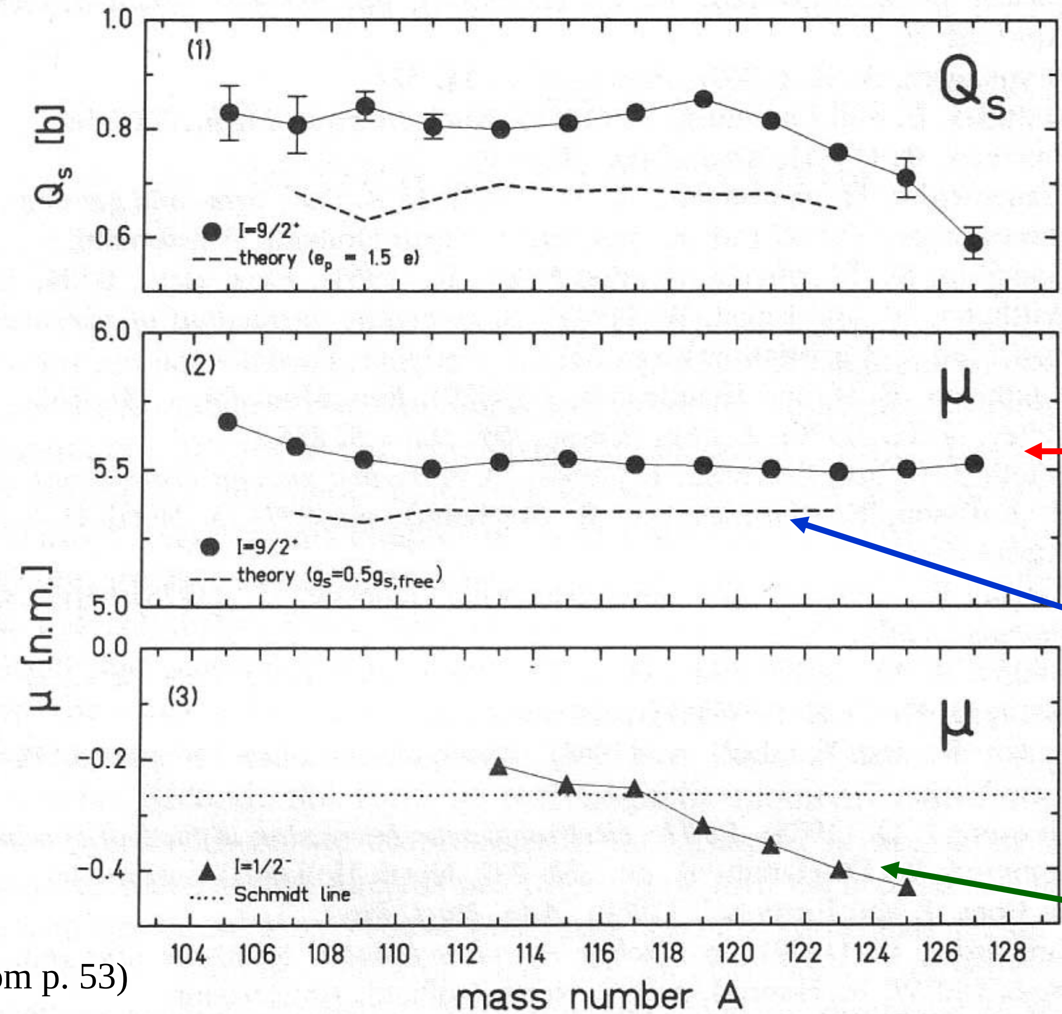
AND

I. S. TOWNER
Chalk River Nuclear Laboratories, Chalk River, Canada

From the preface:

Nuclear moments have had a long history which parallels the development of nuclear physics as a whole. In 1937, Schmidt proposed a standard for magnetic moments based on the assumption that the free-nucleon magnetism persists in heavy nuclei. Then in the early 1950s attempts were made to explain deviations from the Schmidt estimates by invoking meson-exchange corrections or by modifying single-particle wavefunctions so as to include many-body effects. The early 1970s saw the emergence of fundamental problems in trying to reconcile quadrupole transitions and moments in light nuclei with shell-model predictions. Thus the study of nuclear moments could prove an excellent pedagogical tool to acquaint oneself with the complexities and elegance of some of the most current and powerful nuclear models.

Even the most sophisticated nuclear models are not completely successful!



(from p. 53)

Fig. 2.1 Nuclear moments in the odd indium isotopes measured with laser spectroscopy techniques: (1) quadrupole moments of $I = 9/2^+$ ground states; (2) magnetic moments of $I = 9/2^+$ ground states; and (3) magnetic moments of $I = 1/2^-$ isomeric states. The broken lines are theoretical calculations based on core-particle coupling models (see Chapters 5 and 7) using effective coupling constants given in the figure. The Schmidt limit is shown as a dotted line.

Indium isotopes, Z = 49

data, for those with $J^\pi = 9/2^+$ ground states

theory, with residual interactions, and g-factors reduced by 50% compared to free nucleons!

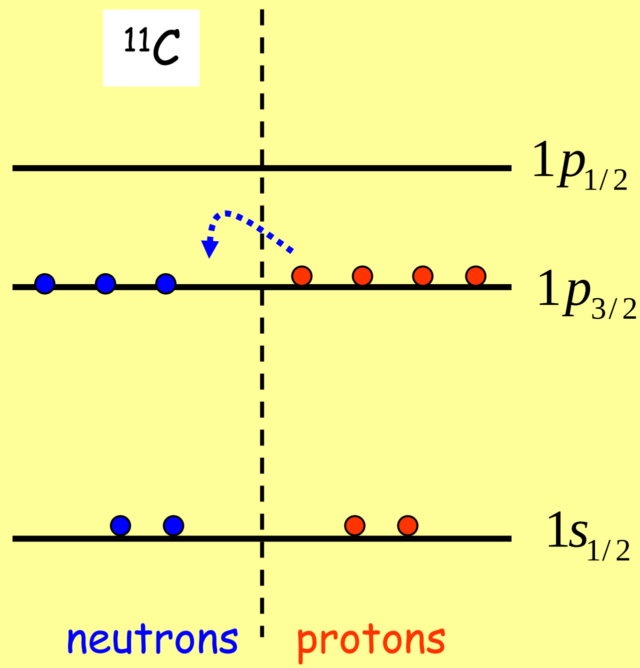
long-lived excited states with $J^\pi = \frac{1}{2}^-$

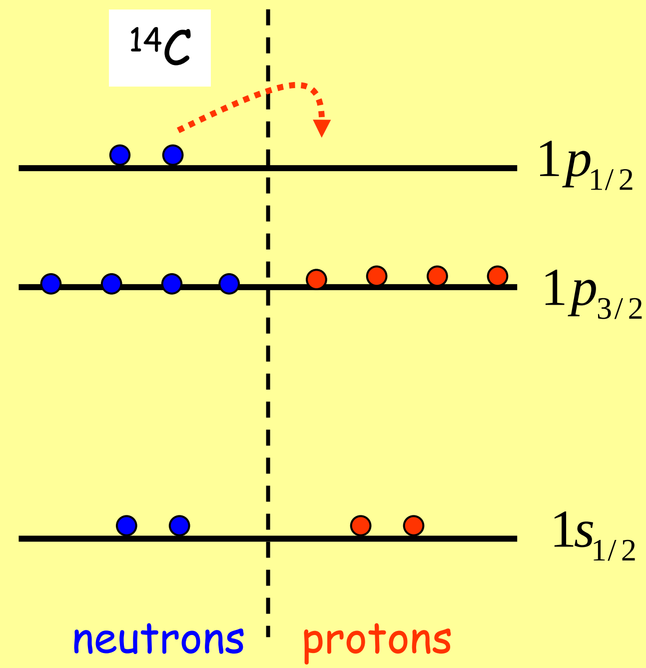
An interesting puzzle from beta decay:

last homework assignment: $^{11}C \rightarrow {}^{11}B + e^{+} + \nu_e$ log(Ft) = 3.6 $t_{\frac{1}{2}}$ = 20.4 min

$^{14}C \rightarrow {}^{14}N + e^{-} + \nu$ log(Ft) = 9.0 $t_{\frac{1}{2}}$ = 5370 yr

why so different? Both decays should have M_{if} = 1 in the shell model (superallowed)





decay is fast, as expected

decay is anomalously slow. – permits the use for ^{14}C dating!

Explanation lies in interactions between nucleons that are not included in the shell model but required to fit experimental details. Even so, ^{14}C is not satisfactorily understood.